On new types of integrable 4-wave interactions

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Abstract.
We start with a Riemann-Hilbert Problems (RHP) with canonical normalization whose sewing functions depends on two or more additional variables. Using Zakharov-Shabat theorem we are able to construct a family of ordinary differential operators for which the solution of the RHP is a common fundamental analytic solution. This family of operators obviously commute provided their coefficients satisfy certain nonlinear evolution equations. Thus we are able to construct new classes of integrable nonlinear evolution equations. We illustrate the method with an example of a new type 4-wave interactions. Its Lax pair consists of operators which are both quadratic in the spectral parameter λ and take values in the $so(5)$ algebra.

Keywords: solitons, N-wave equations, N-wave interactions
PACS: 37K20, 35Q51, 74J30, 78A60

1. INTRODUCTION

The N-wave interactions describe special types of wave-wave interactions [32, 1, 29, 17, 4, 24, 20] which play important role in various fields in physics.

The topic quickly attracted mathematicians from spectral theory, dynamical systems, Lie algebras, Hamiltonian dynamics, differential geometry, see [29, 26, 33, 34, 5, 17, 4] and the numerous references therein. It attracted also a number of physicists because they found important applications of these nonlinear evolution equations (NLEE) in fluid mechanics, nonlinear optics, superconductivity, plasma physics etc. As a result many different approaches for investigating the soliton equations and constructing their Lax representations, soliton solutions, integrals of motion, Hamiltonian hierarchies etc. were developed, see [35, 26, 2, 29, 15, 24, 34, 22].

In the present paper, which is a natural extension of [8], we propose an alternative approach to the same class of equations using as a starting point the Riemann-Hilbert problem (RHP) [37, 38, 33, 34, 29, 36, 23]; the importance of the canonical normalization of RHP was noticed in [11, 6]. Our aim is to show that this allows one to construct rings of commuting operators and in addition gives a tool to study their spectral properties.

In Section 2 below we start with some preliminaries concerning the RHP and their reductions. In Section 3 we use jets of order 1 to reproduce well known results about the 4-wave equations, see Chapter 3 of [29]. In Section 4 we use jets of order 2 which allows us to construct new types of integrable 4-wave interactions whose interaction terms contain quadratic and cubic nonlinearities, as well as x-derivatives. The last Section contains discussion and conclusions.

2. PRELIMINARIES

2.1. RHP with canonical normalization

Let us formulate the RHP:

$$\xi^+(x,t,\lambda) = \xi^-(x,t,\lambda)G(x,t,\lambda), \qquad \lambda^k \in \mathbb{R}, \qquad \lim_{\lambda\to\infty} \xi^+(x,t,\lambda) = \mathbb{1}, \tag{1}$$

where $\xi^{\pm}(x,t,\lambda)$ take values in the simple Lie group $\mathfrak{G}$ with Lie algebra $\mathfrak{g}$. $\xi^+(x,t,\lambda)$ (resp. $\xi^-(x,t,\lambda)$) is an analytic functions of λ for $\operatorname{Im}\lambda^k > 0$ (resp. for $\operatorname{Im}\lambda^k < 0$). For simplicity we consider particular type of dependence of the sewing function $G(x,t,\lambda)$ on the auxiliary variables:

$$i\frac{\partial G}{\partial x} - \lambda^k[J, G(x,t,\lambda)] = 0, \qquad i\frac{\partial G}{\partial t} - \lambda^k[K, G(x,t,\lambda)] = 0. \tag{2}$$

where $k \geq 1$ is a fixed integer and J, K are linearly independent elements of the Cartan subalgebra $J, K \in \mathfrak{h} \subset \mathfrak{g}$.

The canonical normalization of the RHP means that we can introduce the asymptotic expansion

$$\xi^{\pm}(x,t,\lambda) = \exp Q(x,t,\lambda), \qquad Q(x,t,\lambda) = \sum_{k=1}^{\infty} Q_k(x,t)\lambda^{-k}. \tag{3}$$

Since $\xi^{\pm}(x,t,\lambda)$ are group elements then all $Q_k(x,t) \in \mathfrak{g}$. However,

$$\mathcal{J}(x,t,\lambda) = \xi^{\pm}(x,t,\lambda) J \hat{\xi}^{\pm}(x,t,\lambda), \qquad \mathcal{K}(x,t,\lambda) = \xi^{\pm}(x,t,\lambda) K \hat{\xi}^{\pm}(x,t,\lambda), \tag{4}$$

belong to the algebra $\mathfrak{g}$ for any J and K from $\mathfrak{g}$. Since K also belongs to the Cartan subalgebra $\mathfrak{h}$, then

$$[\mathcal{J}(x,t,\lambda), \mathcal{K}(x,t,\lambda)] = 0. \tag{5}$$

An important tool in our considerations plays the well known Zakharov-Shabat theorem [37, 38] formulated below

Theorem 1. *Let $\xi^{\pm}(x,t,\lambda)$ be solutions to the RHP (1) whose sewing function depends on the auxiliary variables x and t via eq. (2). Then $\xi^{\pm}(x,t,\lambda)$ are fundamental solutions of the following set of differential operators:*

$$\begin{aligned} L\xi^{\pm} &\equiv i\frac{\partial \xi^{\pm}}{\partial x} + U(x,t,\lambda)\xi^{\pm}(x,t,\lambda) - \lambda^k [J, \xi^{\pm}(x,t,\lambda)] = 0, \\ M\xi^{\pm} &\equiv i\frac{\partial \xi^{\pm}}{\partial t} + V(x,t,\lambda)\xi^{\pm}(x,t,\lambda) - \lambda^k [K, \xi^{\pm}(x,t,\lambda)] = 0. \end{aligned} \tag{6}$$

Proof. The proof follows the lines of [37, 38]. We introduce the functions:

$$\begin{aligned} g^{\pm}(x,t,\lambda) &= i\frac{\partial \xi^{\pm}}{\partial x}\hat{\xi}^{\pm}(x,t,\lambda) + \lambda^k \xi^{\pm}(x,t,\lambda) J \hat{\xi}^{\pm}(x,t,\lambda), \\ k^{\pm}(x,t,\lambda) &= i\frac{\partial \xi^{\pm}}{\partial t}\hat{\xi}^{\pm}(x,t,\lambda) + \lambda^k \xi^{\pm}(x,t,\lambda) K \hat{\xi}^{\pm}(x,t,\lambda), \end{aligned} \tag{7}$$

where $\hat{\xi}^{\pm} \equiv (\xi^{\pm})^{-1}$, and using (2) prove that

$$g^{+}(x,t,\lambda) = g^{-}(x,t,\lambda), \qquad k^{+}(x,t,\lambda) = k^{-}(x,t,\lambda), \tag{8}$$

which means that these functions are analytic functions of λ in the whole complex λ-plane. Next we find that:

$$\lim_{\lambda\to\infty} g^{+}(x,t,\lambda) = \lambda^k J, \qquad \lim_{\lambda\to\infty} k^{+}(x,t,\lambda) = \lambda^k K. \tag{9}$$

and make use of Liouville theorem to get

$$\begin{aligned} g^{+}(x,t,\lambda) = g^{-}(x,t,\lambda) &= \lambda^k J - \sum_{l=1}^{k} U_l(x,t)\lambda^{k-l}, \\ k^{+}(x,t,\lambda) = k^{-}(x,t,\lambda) &= \lambda^k K - \sum_{l=1}^{k} V_l(x,t)\lambda^{k-l}. \end{aligned} \tag{10}$$

We shall see below that the coefficients $U_l(x,t)$ and $V_l(x,t)$ can be expressed in terms of the asymptotic coefficients Q in eq. (3). □

Lemma 1. *The operators L and M commute:*

$$i\frac{\partial U}{\partial t} - i\frac{\partial V}{\partial x} + [U(x,t,\lambda) - \lambda^k J, V(x,t,\lambda) - \lambda^k K] = 0. \tag{11}$$

where

$$U(x,t,\lambda) = \sum_{l=1}^{k} U_l(x,t)\lambda^{k-l}, \qquad V(x,t,\lambda) = \sum_{l=1}^{k} V_l(x,t)\lambda^{k-l}. \tag{12}$$

Proof. The operators L and M (6) have common fundamental analytic solutions (FAS), i.e. they must commute. The eqs. (11) are an immediate consequence of (6). □

2.2. Jets of order k

In what follows we will consider the jets of order k of $\mathcal{J}(x,\lambda)$ and $\mathcal{K}(x,\lambda)$, see (5). We introduce them by:

$$\begin{aligned}\mathcal{J}(x,t,\lambda) &\equiv \left(\lambda^k \xi^{\pm}(x,t,\lambda) J_l \hat{\xi}^{\pm}(x,t,\lambda)\right)_+ = \lambda^k J - U(x,t,\lambda),\\ \mathcal{K}(x,t,\lambda) &\equiv \left(\lambda^k \xi^{\pm}(x,t,\lambda) K \hat{\xi}^{\pm}(x,t,\lambda)\right)_+ = \lambda^k K - V(x,t,\lambda).\end{aligned} \tag{13}$$

The subscript $+$ used above means that we insert the asymptotic expansions of $\xi^{\pm}$ and their inverse (3) and cut off the terms with negative powers of λ.

Obviously $U(x) \in \mathfrak{g}$ can be expressed in terms of $Q(x)$. In doing this we take into account (5) and obtain [19]

$$\mathcal{J}(x,t,\lambda) = J + \sum_{k=1}^{\infty} \frac{1}{k!} \mathrm{ad}_Q^k J, \qquad \mathcal{K}(x,t,\lambda) = K + \sum_{k=1}^{\infty} \frac{1}{k!} \mathrm{ad}_Q^k K, \tag{14}$$

and therefore for U_l we get:

$$\begin{aligned}
&U_1(x,t) = -\mathrm{ad}_{Q_1} J, \qquad U_2(x,t) = -\mathrm{ad}_{Q_2} J - \frac{1}{2}\mathrm{ad}_{Q_1}^2 J\\
&U_3(x,t) = -\mathrm{ad}_{Q_3} J - \frac{1}{2}\left(\mathrm{ad}_{Q_2}\mathrm{ad}_{Q_1} + \mathrm{ad}_{Q_1}\mathrm{ad}_{Q_2}\right) J - \frac{1}{6}\mathrm{ad}_{Q_1}^3 J\\
&\qquad \vdots\\
&U_k(x,t) = -\mathrm{ad}_{Q_k} J - \frac{1}{2}\sum_{s+p=k} \mathrm{ad}_Q \mathrm{ad}_{Q_p} J - \frac{1}{6}\sum_{s+p+r=k} \mathrm{ad}_Q \mathrm{ad}_{Q_p} \mathrm{ad}_{Q_r} J - \cdots - \frac{1}{k!}\mathrm{ad}_{Q_1}^k J,
\end{aligned} \tag{15}$$

and similar expressions for $V_l(x,t)$ with J replaced by K.

2.3. Reductions of polynomial bundles

An important tool to construct new integrable NLEE is based on Mikhailov's group of reductions [27]. Below we will use mainly $\mathbb{Z}_2$ and $\mathbb{Z}_N$ with $N > 2$ reduction groups. The basic $\mathbb{Z}_2$-examples are as follows:

$$\begin{aligned}
&\text{a)} \qquad A\xi^{+,\dagger}(x,t,\varepsilon\lambda^*)\hat{A} = \hat{\xi}^-(x,t,\lambda), \qquad AQ^{\dagger}(x,t,\varepsilon\lambda^*)\hat{A} = -Q(x,t,\lambda),\\
&\text{b)} \qquad B\xi^{+,*}(x,t,\varepsilon\lambda^*)\hat{B} = \xi^-(x,t,\lambda), \qquad BQ^{*}(x,t,\varepsilon\lambda^*)\hat{B} = Q(x,t,\lambda),\\
&\text{c)} \qquad C\xi^{+,T}(x,t,-\lambda)\hat{C} = \hat{\xi}^-(x,t,\lambda), \qquad CQ^{\dagger}(x,t,-\lambda)\hat{C} = -Q(x,t,\lambda),
\end{aligned} \tag{16}$$

where $\varepsilon^2 = 1$ and A, B and C are elements of the group $\mathfrak{G}$ such that $A^2 = B^2 = C^2 = \mathbb{1}$. As for the $\mathbb{Z}_N$-reductions we may have:

$$D\xi^{\pm}(x,t,\omega\lambda)\hat{D} = \xi^{\pm}(x,t,\lambda), \qquad DQ(x,t,\omega\lambda)\hat{D} = Q(x,t,\lambda), \tag{17}$$

where $\omega^N = 1$ and $D^N = \mathbb{1}$.

These relations allow us to introduce algebraic relations between the matrix elements of $Q(x,t,\lambda)$ which will be automatically compatible with the NLEE. The classes of inequivalent reductions of the N-wave equations related to the low-rank simple Lie algebras are given in [9, 10, 12, 13, 13, 14, 16].

3. THE CLASSICAL 4-WAVE INTERACTIONS

The 4-wave interactions were discovered by Zakharov (see Chapter 3 of [29]) who used a 4×4 Lax pair with an additional reduction which effectively reduced it to the subalgebra $sp(4)$. It is well known that $sp(4)$ is isomorphic to $so(5)$, so it is a matter of taste which one to choose. We remind that the root system Δ of $so(5)$ has 8 roots (see [19]):

$$\Delta \equiv \Delta_+ \cup \Delta_-, \qquad \Delta_{\pm} = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(\alpha_1+2\alpha_2)\}, \tag{18}$$

where $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_2$ are the two simple roots.

With this notations the Lax pair proposed by Zakharov takes the usual form for the N-wave equations

$$\begin{aligned} L\psi &= i\frac{\partial \psi}{\partial x} + (U_1(x,t) + \lambda J)\psi(x,t,\lambda) = 0, \\ M\psi &= i\frac{\partial \psi}{\partial t} + (V_1(x,t) + \lambda K)\psi(x,t,\lambda) = 0, \end{aligned} \tag{19}$$

where the potentials U_1 and V_1 are of the form:

$$\begin{aligned} U_1(x,t) &= [J, Q_1(x,t)], \qquad\qquad V_1(x,t) = [K, Q_1(x,t)], \\ Q_1(x,t) &= \sum_{\alpha\in\Delta_+} (q_\alpha(x,t)E_\alpha + p_\alpha(x,t)E_{-\alpha}) \end{aligned} \tag{20}$$

We impose the natural reduction (16a) which provide $p_\alpha = \varepsilon q_\alpha^*$. Thus the 4-wave equations take the form

$$i\left[J, \frac{\partial Q_1}{\partial t}\right] - i\left[K, \frac{\partial Q_1}{\partial x}\right] + [[J,Q_1],[K,Q_1(x,t)]] = 0, \tag{21}$$

or in components:

$$\begin{aligned} 2i(a_1-a_2)\frac{\partial u_1}{\partial t} - 2i(b_1-b_2)\frac{\partial u_1}{\partial x} + 2\kappa v_2 u_4 &= 0, \\ 2ia_2\frac{\partial u_2}{\partial t} - 2ib_2\frac{\partial u_2}{\partial x} + 2\kappa(v_4u_3 + u_4v_1) &= 0, \\ 2ia_1\frac{\partial u_4}{\partial t} - 2ib_1\frac{\partial u_4}{\partial x} + 2\kappa(v_2u_3 - u_2u_1) &= 0, \\ 2i(a_1+a_2)\frac{\partial u_3}{\partial t} - 2i(b_1+b_2)\frac{\partial u_3}{\partial x} - 2\kappa u_2u_4 &= 0, \end{aligned} \tag{22}$$

where $u_1 = q_{\alpha_1}$, $u_2 = q_{\alpha_2}$, $u_4 = q_{\alpha_1+\alpha_2}$ and $u_3 = q_{\alpha_1+2\alpha_2}$.

4. NEW TYPES OF 4-WAVE INTERACTIONS – AN EXAMPLE

Here we will give an example of a new type of integrable 4-wave equations.

The Lax pair for these new equations will be provided by:

$$\begin{aligned} L\psi &= i\frac{\partial \psi}{\partial x} + (U_2(x,t) + \lambda U_1(x,t) - \lambda^2 J)\psi(x,t,\lambda) = 0, \\ M\psi &= i\frac{\partial \psi}{\partial t} + (V_2(x,t) + \lambda V_1(x,t) - \lambda^2 K)\psi(x,t,\lambda) = 0, \end{aligned} \tag{23}$$

where $U_j(x,t)$ and $V_j(x,t)$ are fast decaying smooth functions taking values in the Lie algebra $so(5)$

$$\begin{aligned} U_1(x,t) &= [J, Q_1(x,t)], \qquad U_2(x,t) = [J, Q_2(x,t)] - \frac{1}{2}\mathrm{ad}_{Q_1}^2 J, \\ V_1(x,t) &= [K, Q_1(x,t)], \qquad V_2(x,t) = [K, Q_2(x,t)] - \frac{1}{2}\mathrm{ad}_{Q_1}^2 K. \end{aligned} \tag{24}$$

Here $\mathrm{ad}_{Q_1} X \equiv [Q_1(x,t), X]$ and J and K are two linearly independent constant elements of the Cartan subalgebra of $so(5)$.

If we assume $Q_1(x,t)$ and $Q_2(x,t)$ to be generic elements of $so(5)$ then they can be parametrized by:

$$\begin{aligned} Q_1(x,t) &= \sum_{\alpha\in\Delta_+} (q_\alpha^1 E_\alpha + p_\alpha^1 E_{-\alpha}) + r_1^1 H_{e_1} + r_2^1 H_{e_2}, \\ Q_2(x,t) &= \sum_{\alpha\in\Delta_+} (q_\alpha^2 E_\alpha + p_\alpha^2 E_{-\alpha}) + r_1^2 H_{e_1} + r_2^2 H_{e_2}, \\ J &= a_1 H_{e_1} + a_2 H_{e_2} = \mathrm{diag}\,(a_1, a_2, 0, -a_2, -a_1), \qquad K = b_1 H_{e_1} + b_2 H_{e_2} = \mathrm{diag}\,(b_1, b_2, 0, -b_2, -b_1), \end{aligned} \tag{25}$$

Next we impose on $Q_1(x,t)$ and $Q_2(x,t)$ the natural reduction (16a)

$$B_0 U(x,t,\varepsilon\lambda^*)^\dagger B_0^{-1} = U(x,t,\lambda), \qquad B_0 = \mathrm{diag}\,(1,\varepsilon,1,\varepsilon,1), \quad \varepsilon^2 = 1. \tag{26}$$

As a result:

$$B_0(\chi^+(x,t,\varepsilon\lambda^*))^\dagger B_0^{-1} = (\chi^-(x,t,\lambda))^{-1}, \qquad B_0(T(t,\varepsilon\lambda^*))^\dagger B_0^{-1} = (T(t,n\lambda))^{-1}, \tag{27}$$

which provide $p^1_\alpha = \varepsilon(q^1_\alpha)^*$, $p^2_\alpha = \varepsilon(q^2_\alpha)^*$. Then eq. (32) will be a (rather complicated) system of 8 NLEE for the 8 independent matrix elements q^1_α and q^2_α.

However we can impose additional $\mathbb{Z}_2$ reduction condition

$$D\xi^\pm(x,t,-\lambda)\hat{D} = \xi^\pm(x,t,\lambda), \qquad DQ(x,t,-\lambda)\hat{D} = Q(x,t,\lambda), \tag{28}$$

where $D = \mathrm{diag}\,(1,-1,1,-1,1)$. More precisely this means:

$$\begin{aligned} Q_1(x,t) &= u_1E_{e_1-e_2} + u_2E_{e_2} + u_3E_{e_1+e_2} + v_1E_{-e_1+e_2} + v_2E_{-e_2} + v_3E_{-e_1-e_2} = \begin{pmatrix} 0 & u_1 & 0 & u_3 & 0 \\ v_1 & 0 & u_2 & 0 & u_3 \\ 0 & v_2 & 0 & u_2 & 0 \\ v_3 & 0 & v_2 & 0 & u_1 \\ 0 & v_3 & 0 & v_1 & 0 \end{pmatrix}, \\ Q_2(x,t) &= u_4E_{e_1} + v_4E_{-e_1} + w_1H_{e_1} + w_2H_{e_2} = \begin{pmatrix} w_1 & 0 & u_4 & 0 & 0 \\ 0 & w_2 & 0 & 0 & 0 \\ w_4 & 0 & 0 & 0 & u_4 \\ 0 & 0 & 0 & -w_2 & 0 \\ 0 & 0 & -v_4 & 0 & -w_1 \end{pmatrix}, \\ J &= a_1H_{e_1} + a_2H_{e_2} = \mathrm{diag}\,(a_1,a_2,0,-a_2,-a_1), \qquad K = b_1H_{e_1} + b_2H_{e_2} = \mathrm{diag}\,(b_1,b_2,0,-b_2,-b_1), \end{aligned} \tag{29}$$

Combining both reductions for the matrix elements of $Q_j(x,t)$ we have:

$$v_1 = \varepsilon u_1^*, \qquad v_2 = \varepsilon u_2^*, \qquad v_3 = \varepsilon u_3^*, \qquad v_4 = u_4^*, \tag{30}$$

The commutativity condition for the Lax pair (23)

$$i\left(\frac{\partial V_2}{\partial x} + \lambda\frac{\partial V_1}{\partial x}\right) - i\left(\frac{\partial U_2}{\partial t} + \lambda\frac{\partial U_1}{\partial t}\right) + [U_2 + \lambda U_1 - \lambda^2 J, V_2 + \lambda V_1 - \lambda^2 K] = 0 \tag{31}$$

must hold identically with respect to λ. One can check, that with the choice (25) for U_i and V_i, $i = 1,2$ the terms proportional to λ^4, λ^3 and λ^2 vanish identically. The term proportional to λ and the λ-independent term vanish provided Q_i satisfy the NLEE:

$$\begin{aligned} &i\frac{\partial V_1}{\partial x} - i\frac{\partial U_1}{\partial t} + [U_2,V_1] + [U_1,V_1] = 0, \\ &i\frac{\partial V_2}{\partial x} - i\frac{\partial U_2}{\partial t} + [U_2,V_2] = 0. \end{aligned} \tag{32}$$

In components the corresponding NLEE:

$$\begin{aligned} &-2i(a_1-a_2)\frac{\partial u_1}{\partial t} + 2i(b_1-b_2)\frac{\partial u_1}{\partial x} + \kappa\varepsilon u_2^*(\varepsilon u_2^* u_3 - u_1u_2 - 2u_4) = 0, \\ &-2ia_2\frac{\partial u_2}{\partial t} + 2ib_2\frac{\partial u_2}{\partial x} - \kappa(u_2\varepsilon(|u_3|^2 - |u_1|^2) + 2u_3u_4^* + 2\varepsilon u_1^* u_4) = 0, \\ &-2i(a_1+a_2)\frac{\partial u_3}{\partial t} + 2i(b_1+b_2)\frac{\partial u_3}{\partial x} + \kappa u_2(\varepsilon u_2^* u_3 - u_1u_2 + 2u_4) = 0, \\ &-2ia_1\frac{\partial u_4}{\partial t} + 2ib_1\frac{\partial u_4}{\partial x} + i\frac{\partial}{\partial t}\left(-(2a_2-a_1)u_1u_2 + (2a_2+a_1)\varepsilon u_2^* u_3\right) + i(2b_2-b_1)\frac{\partial(u_1u_2)}{\partial x} \\ &-i(2b_2+b_1)\varepsilon\frac{\partial(u_2^* u_3)}{\partial x} - \kappa\left(2\varepsilon u_4(|u_1|^2 - |u_3|^2) + \varepsilon u_1u_2(|u_1|^2 + 3|u_3|^2) - u_3u_2^*(3|u_1|^2 + |u_3|^2)\right) = 0. \end{aligned} \tag{33}$$

Let us now introduce

$$U_4 = u_4 - \frac{1}{2a_1}((a_1-a_2)u_1u_2 + (a_1+a_2)\varepsilon u_3 u_2^*). \tag{34}$$

As a result we get:

$$\begin{aligned}
&-2i(a_1-a_2)\frac{\partial u_1}{\partial t} + 2i(b_1-b_2)\frac{\partial u_1}{\partial x} - \frac{\kappa\varepsilon}{a_1} u_2^*(2a_1U_4 + \varepsilon a_2 u_2^* u_3 + (2a_1-a_2)u_1u_2) = 0,\\
&\quad -2ia_2\frac{\partial u_2}{\partial t} + 2ib_2\frac{\partial u_2}{\partial x} - \frac{\kappa\varepsilon}{a_1}u_2((2a_1+a_2)|u_3|^2 - a_2|u_1|^2) - 2\kappa(u_3U_4^* + \varepsilon u_1^* U_4 + u_1^* u_2^* u_3) = 0,\\
&-2i(a_1+a_2)\frac{\partial u_3}{\partial t} + 2i(b_1+b_2)\frac{\partial u_3}{\partial x} + \frac{\kappa}{a_1}u_2(\varepsilon(2a_1+a_2)u_2^*u_3 - a_2u_1u_2 + 2a_1U_4) = 0,\\
&\quad -2ia_1\frac{\partial U_4}{\partial t} + 2ib_1\frac{\partial U_4}{\partial x} + \frac{i\kappa}{a_1}\frac{\partial u_1u_2}{\partial x} - \frac{i\kappa\varepsilon}{a_1}\frac{\partial u_2^*u_3}{\partial x}\\
&\qquad\qquad - \frac{\kappa}{a_1}\left(2\varepsilon U_4(|u_1|^2 - |u_3|^2) + (\varepsilon u_1u_2 - u_3u_2^*)((2a_1-a_2)|u_1|^2 + (2a_1+a_2)|u_3|^2)\right) = 0,
\end{aligned} \tag{35}$$

5. DISCUSSION AND CONCLUSIONS

On the example of RHP for functions taking values in the $SO(5)$-group we have outlined the construction of families of commuting operators. Applied to jets of order 1 with this method reproduces the well known results for the 4-wave equations.

Using jets of order 2 gives us the simplest nontrivial examples for new types of integrable 4-wave equation whose interaction terms contain quadratic and cubic nonlinearities, as well as x-derivatives. These equations also allow integrable extensions to three-dimensional space-time, see [8].

It is not difficult to obtain other new integrable 4-wave equations. To this end we can use: i) different power k of the polynomials $U(x,t,\lambda)$ and $V(x,t,\lambda)$ ii) different types of grading; and iii) different reductions of U and V.

For example another integrable 4-wave interactions model can be obtained using Lax pair of degree 4 with respect to λ:

$$\begin{aligned}
L\psi &= i\frac{\partial\psi}{\partial x} + (U_4(x,t) + \lambda U_3(x,t) + \lambda^2 U_2(x,t) + \lambda^3 U_1(x,t) - \lambda^4 J)\psi(x,t,\lambda) = 0,\\
M\psi &= i\frac{\partial\psi}{\partial t} + (V_4(x,t) + \lambda V_3(x,t) + \lambda * 2V_2(x,t) + \lambda^3 V_1(x,t) - \lambda^4 K)\psi(x,t,\lambda) = 0,
\end{aligned} \tag{36}$$

where $U_j(x,t)$ and $V_j(x,t)$ for $j=1,2$ are again as in (24); for $j=3$ and $j=4$ we have

$$\begin{aligned}
U_3(x,t) &= [J, Q_3(x,t)] - \frac{1}{2}(\mathrm{ad}_{Q_1}\mathrm{ad}_{Q_2} + \mathrm{ad}_{Q_2}\mathrm{ad}_{Q_1})J + \frac{1}{6}\mathrm{ad}^3_{Q_1}J,\\
V_3(x,t) &= [K, Q_3(x,t)] - \frac{1}{2}(\mathrm{ad}_{Q_1}\mathrm{ad}_{Q_2} + \mathrm{ad}_{Q_2}\mathrm{ad}_{Q_1})K + \frac{1}{6}\mathrm{ad}^3_{Q_1}K,\\
U_4(x,t) &= [J, Q_4(x,t)] - \frac{1}{2}\sum_{i+j=4}\mathrm{ad}_{Q_i}\mathrm{ad}_{Q_j}J - \frac{1}{6}\sum_{i+j+k=4}\mathrm{ad}_{Q_i}\mathrm{ad}_{Q_j}\mathrm{ad}_{Q_k}J - \frac{1}{24}\mathrm{ad}^4_{Q_1}J,\\
U_4(x,t) &= [K, Q_4(x,t)] - \frac{1}{2}\sum_{i+j=4}\mathrm{ad}_{Q_i}\mathrm{ad}_{Q_j}K - \frac{1}{6}\sum_{i+j+k=4}\mathrm{ad}_{Q_i}\mathrm{ad}_{Q_j}\mathrm{ad}_{Q_k}K - \frac{1}{24}\mathrm{ad}^4_{Q_1}K,
\end{aligned} \tag{37}$$

To keep the number of the independent functions down to 4 we will use in addition to (16a), an additional $\mathbb{Z}_4$-reduction of the form (17) which leads to the following parametrization of Q_j:

$$\begin{aligned}
&Q_1(x,t) = u_1E_{e_1-e_2} + u_2E_{e_2} + v_1E_{-e_1+e_2} + v_2E_{-e_2}, && Q_2(x,t) = u_3E_{e_1-e_2} + v_3E_{-e_2},\\
&Q_3(x,t) == u_4E_{e_1+e_2} + v_4E_{-e_1-e_2}, && Q_4(x,t) == w_4H_{e_1} + z_4H_{e_2},
\end{aligned} \tag{38}$$

It jets $U(x,t,\lambda)$ can be view as elements of special co-adjoint orbits of the relevant Kac-Moody algebra, generated by the chosen grading of $\mathfrak{f}$. This allows one to define a hierarchy of Poisson brackets, see [25, 30, 31], which along with the conservation laws will provide the hierarchy of Hamiltonian structures of these NLEE.

Finally we can also apply Zakharov-Shabat dressing method [37, 38, 28, 21] for constructing their explicit (N-soliton) solutions. Instead of solving the inverse scattering problem for L we would rather deal with a Riemann-Hilbert problem with canonical normalization. For polynomials of order k the contour on which the RHP is defined consists of k straight lines $l_k:\ \arg\lambda = \pi i/k$ passing through the origin. Of course, it may necessary to use dressing factors with more specific λ-dependence.

This approach can be used also to analyze the NLEE derived by Gel'fand-Dickey approach [3, 18]. It would provide the possibility to systematically construct the spectral decompositions that linearize the relevant NLEE [7, 17]. Still more challenging is to study the soliton interactions of the new types of 4- and N-wave equations.

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